Linear Programs and Applications to Approximation Algorithms

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Outline

1. Linear Programs
   - Understanding an LP
   - The notion of LP Duality (ref: Approximation Algorithms by Vijay V. Vazirani)
   - Weak Duality Theorem
   - Strong Duality or LP Duality Theorem
   - Complementary Slackness
2. Approximation Algorithms via LP/ LP Duality
   - LP Rounding
3. Approximation Algorithms via LP/ LP Duality
   - Primal-Dual

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\begin{aligned}
\textit{Maximize} \quad z = 2x_1 + x_2 & & \\
\text{s.t.} \qquad -3x_1 + 2x_2 & \leq & 6 \\
x_1 + x_2 & \leq & 8 \\
2x_1 - x_2 & \leq & 4 \\
x_1, x_2 & \geq & 0
\end{aligned}
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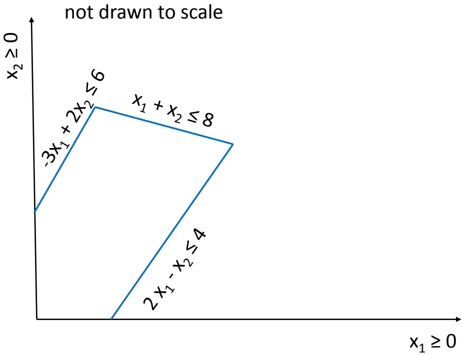

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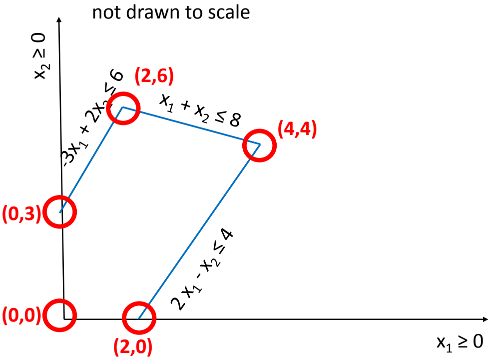

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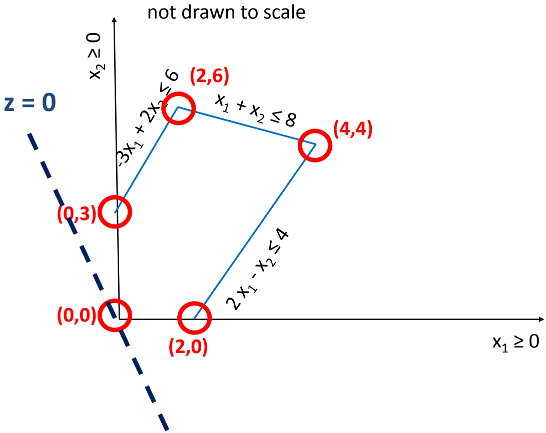

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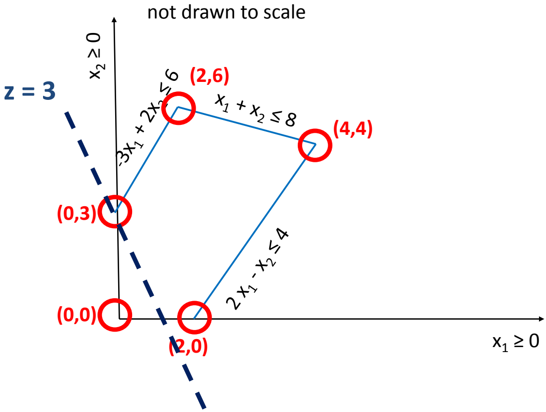

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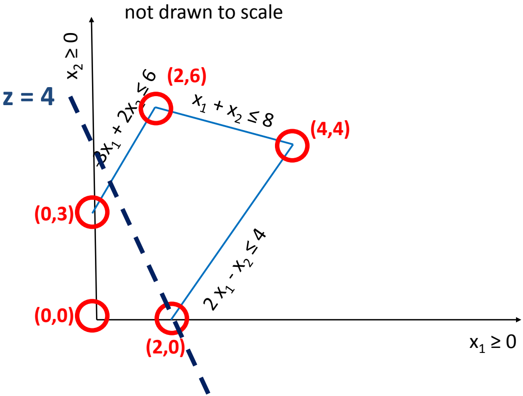

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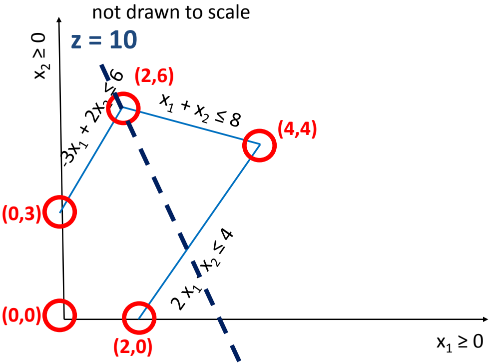

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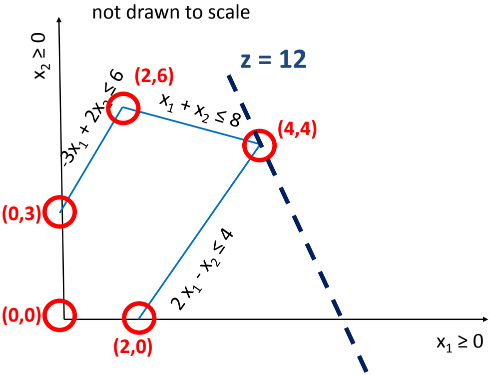

Outline

LP Duality - the notion

$$\begin{array}{lrcl} \textit{Minimize} & 7x_1 + x_2 + 5x_3 & & \\ \text{s.t.} & x_1 - x_2 + x_3 & \geq & 10 \\ & 5x_1 + 2x_2 - x_3 & \geq & 6 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

How do you exhibit an upper bound to the OPT?

Consider the feasible solution $X =< 2, 1, 3 >$

objective value at X = 7*2 + 1*1 + 5*3 = 30

30 is an upper bound

LP Duality - the notion

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How do you exhibit an upper bound to the OPT?

For a minimization problem:
objective value for any feasible solution
is an upper bound

LP Duality - the notion

$$\begin{aligned} \textit{Minimize} \quad & \mathbf{7x_1 + x_2 + 5x_3} \\ \text{s.t.} \quad & \mathbf{x_1 - x_2 + x_3} \geq 10 \\ & 5x_1 + 2x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

What about a lower bound to the OPT?

$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + x_3$ Why?

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What about a lower bound to the OPT?

$$\mathbf{7x_1 + x_2 + 5x_3} \geq x_1 - x_2 + x_3 \geq \mathbf{10}$$

A better lower bound

$$\mathbf{7x_1 + x_2 + 5x_3} \geq (x_1 - x_2 + x_3) + (5x_1 + 2x_2 - x_3) \geq \mathbf{10 + 6 = 16}$$

LP Duality - the notion

$$\begin{array}{lrcl} \textit{Minimize} & \mathbf{7x_1 + x_2 + 5x_3} & & \\ \text{s.t.} & \mathbf{x_1 - x_2 + x_3} & \geq & 10 \\ & \mathbf{5x_1 + 2x_2 - x_3} & \geq & 6 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

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LP Duality - the notion

$$\begin{array}{rrcll} \textit{Minimize} & \mathbf{7x_1 + x_2 + 5x_3} & & & \\ \text{s.t.} & \mathbf{x_1 - x_2 + x_3} & \geq & 10 & y_1 \\ & \mathbf{5x_1 + 2x_2 - x_3} & \geq & 6 & y_2 \\ & x_1, x_2, x_3 & \geq & 0 & \end{array}$$

- assign a non-negative coefficient y_i to every constraint such that $7x_1 + x_2 + 5x_3 \geq y_1(x_1 - x_2 + x_3) + y_2(5x_1 + 2x_2 - x_3)$
- lower bound is $10y_1 + 6y_2$

LP Duality - the notion

The problem of finding the best lower bound can be formulated as a linear program

Primal

$$\begin{aligned} \textit{Minimize} \quad & 7x_1 + x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 \geq 10 \\ & 5x_1 + 2x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \textit{Maximize} \quad & 10y_1 + 6y_2 \\ \text{s.t.} \quad & y_1 + 5y_2 \leq 7 \\ & -y_1 + 2y_2 \leq 1 \\ & 3y_1 - y_2 \leq 5 \\ & y_1, y_2 \geq 0 \end{aligned}$$

LP Duality - the notion

Primal

$$\begin{aligned} \textit{Minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j &\geq b_i \quad \forall i \\ x_j &\geq 0 \quad \forall j \end{aligned}$$

Dual

$$\begin{aligned} \textit{Maximize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i &\leq c_j \quad \forall j \\ y_i &\geq 0 \quad \forall i \end{aligned}$$

$$\begin{aligned} \textit{Minimize} \quad & c^T x \\ \text{s.t.} \quad Ax &\geq b \\ x &\geq 0 \end{aligned}$$

$$\begin{aligned} \textit{Maximize} \quad & b^T y \\ \text{s.t.} \quad A^T y &\leq c \\ y &\geq 0 \end{aligned}$$

LP Duality - the notion

Dual

Dual of Dual

$$\begin{array}{lrcl} \textit{Maximize} & & & b^T y \\ \text{s.t.} & A^T y & \leq & c \\ & y & \geq & 0 \end{array}$$

What is the dual of the dual ?

LP Duality - the notion

Dual

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What is the dual of the dual ? **Primal**

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LP Duality - Weak Duality Theorem

Weak Duality: *If* $\mathbf{x} = < x_1, x_2, ..., x_n >$ *and* $\mathbf{y} = < y_1, y_2, ..., y_m >$ *are feasible solutions for the primal and dual program, respectively, then:*

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof:

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j &\geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \\ &\geq \sum_{i=1}^{m} b_i y_i \end{aligned}$$

LP Duality - Weak Duality Theorem

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$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof:

$$\begin{aligned}
\sum_{j=1}^{n} c_j x_j &\geq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j \\
&= \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \\
&\geq \sum_{i=1}^{m} b_i y_i
\end{aligned}$$

Outline

Strong Duality:

- *The primal program has finite optimum iff its dual has finite optimum*
- *if* $\mathbf{x}^* =< x_1^*, x_2^*, \ldots\ldots, x_n^* >$ *and* $\mathbf{y}^* =< y_1^*, y_2^*, \ldots\ldots, y_m^* >$ *are optimal solutions for the primal and dual programs, respectively, then:*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Outline

LP Duality- Complementary Slackness

Complementary Slackness Conditions: *If* **x** *and* **y** *are respectively feasible solutions for primal and dual, then,*

x *and* **y** *are optimal iff:*

- **Primal:** $\forall\ 1 \leq j \leq n$: *either* $x_j = 0$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$
- **Dual:** $\forall\ 1 \leq i \leq m$: *either* $y_i = 0$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$

LP Duality- Complementary Slackness

Proof: By Strong Duality Theorem:

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i = \sum_{i=1}^{m} b_i y_i$$

Thus $\sum_{j=1}^{n} (c_j - \sum_{i=1}^{m} a_{ij} y_i) x_j = 0$

So either $x_j = 0$ or $c_j - \sum_{i=1}^{m} a_{ij} y_i = 0$

Thus $x_j > 0$ implies $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Outline

Approximation Algorithms

Vertex cover problem:

Given: Graph $G = (V, E)$

To find: A subset $S \subset V$ of minimum cardinality such that for every edge $(u, v) \in E$, either $u \in S$ or $v \in S$, or both

Integer Program

$$\begin{aligned} \textit{Minimize} \quad & \sum_{u: u \in V} x_u \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall \quad (u, v) \in E \\ & x_u \in \{0, 1\} \quad \forall \quad u \in V \end{aligned}$$

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Approximation Algorithms: LP Rounding

An LP rounding based approximation algorithm for vertex cover:

1. Solve the relaxed Linear Program corresponding to the given problem:

$$\begin{array}{lrl} \textit{Minimize} & \sum_{u:u\in V} x_u & \\ \text{s.t.} & x_u + x_v \geq 1 & \forall \quad (u,v) \in E \\ & 0 \leq x_u \leq 1 & \forall \quad u \in V \end{array}$$

2. $S = \{u \in V : x_u \geq \frac{1}{2}\}$

Claim 1: S is a feasible solution

- Consider an edge (u, v)
- $x_u + x_v \geq 1$
- $max\{x_u, x_v\} \geq \frac{1}{2}$
- hence at least one of u and v is picked in S

Approximation Algorithms: LP Rounding

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Claim 2: $|S| \leq 2 \cdot OPT$

$$
\begin{aligned}
|S| &\leq \sum_{u \in S} 2 \cdot x_u \quad (since \;\; x_u \geq \frac{1}{2} \;\forall\, u \in S) \\
&= 2 \sum_{u \in S} x_u \\
&\leq 2 \cdot OPT
\end{aligned}
$$

Approximation Algorithms: LP Rounding

Claim 2: $|S| \leq 2 \cdot OPT$

$$\begin{aligned} |S| &\leq \sum_{u \in S} 2 \cdot x_u \quad \left(\mathit{since}\ \ x_u \geq \frac{1}{2}\ \ \forall\ u \in S\right) \\ &= 2 \sum_{u \in S} x_u \\ &\leq 2 \cdot OPT \end{aligned}$$

Approximation Algorithms: LP Rounding

Theorem: *The LP-rounding algorithm is a 2-approximation algorithm for the vertex cover problem*

Proof:

Claim 1 and Claim 2 imply the theorem

Outline

Approximation Algorithms: Primal-Dual for VC

Primal LP:

$$\begin{aligned} &\textit{Minimize} \quad \sum_{u:u\in V} x_u \\ &\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall \quad e=(u,v)\in E \\ &\qquad\quad 0 \leq x_u \leq 1 \quad \forall \quad u \in V \end{aligned}$$

Dual LP:

$$\begin{aligned} &\textit{Maximize} \quad \sum_{e:e\in E} y_e \\ &\text{s.t.} \quad \sum_{e:e\sim u} y_e \leq 1 \quad \forall \quad u\in V \\ &\qquad\quad 0 \leq y_e \leq 1 \quad \forall \quad e \in E \end{aligned}$$

Complementary Slackness Conditions:

1. **Primal:** either $x_u = 0$ or $\sum_{e:e\sim u} y_e = 1 \quad \forall u \in V$
2. **Dual:** either $y_e = 0$ or $x_u + x_v = 1 \quad \forall e = (u, v) \in E$

Approximation Algorithms: Primal-Dual for VC

Primal-Dual Schema applied to the Vertex Cover problem:

1. Let $\mathbf{x}$ and $\mathbf{y}$ denote solutions to Primal and Dual respectively. Start with $\mathbf{x} = 0$ and $\mathbf{y} = 0$.
   solution to Primal VC $\mathbf{x} = 0$; solution to Dual VC $\mathbf{y} = 0$
   observe that $\mathbf{y}$ is dual feasible but $\mathbf{x}$ is not primal feasible

2. Until the Primal is feasible:
   - raise the dual variables (either simultaneously or one-by-one) while maintaining the dual feasibility
     raise $y_e \; \forall e \in E$ until some dual constraint goes tight
   - for every tight dual constraint, *freeze* the value of y and raise the corresponding x
     $\forall u$ s.t. $\sum_{e:e\sim u} y_e = 1$, set $x_u = 1$; delete all edges incident on u

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   solution to Primal VC **x** = 0; solution to Dual VC **y** = 0
   observe that **y** is dual feasible but **x** is not primal feasible

2. Until the Primal is feasible:
   - raise the dual variables (either simultaneously or one-by-one) while maintaining the dual feasibility
     raise $y_e \; \forall e \in E$ until some dual constraint goes tight
   - for every tight dual constraint, *freeze* the value of y and raise the corresponding x
     $\forall u$ s.t. $\sum_{e:e\sim u} y_e = 1$, set $x_u = 1$; delete all edges incident on u

Approximation Algorithms: Primal-Dual for VC

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- we ensure feasibility of **y** at every step; so **y** is feasible at the end
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Approximation Algorithms: Primal-Dual for VC

Claim 2: x and y satisfy Primal Complementary Slackness

recall PCS: either $x_u = 0$ or $\sum_{e:e\sim u} y_e = 1 \quad \forall u \in V$

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Claim 3: x and y 2-approximate Dual Complementary Slackness

recall DCS: either $y_e = 0$ or $x_u + x_v = 1 \quad \forall e = (u, v) \in E$

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Claim 4: If x and y are feasible for Primal and Dual, satify PCS and α-approximate DCS, then:

$$cost_{primal}(x) \leq \alpha \cdot cost_{dual}(y)$$

$$\begin{aligned} cost_{primal}(x) = \sum_i x_i &= \sum_i (\sum_j a_{ij} y_j) x_i \\ &= \sum_j (\sum_i a_{ij} x_i) y_j \\ &\leq \sum_j \alpha y_j \\ &= \alpha \sum_j y_j = \alpha cost_{dual}(y) \end{aligned}$$

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Approximation Algorithms: Primal-Dual for VC

Theorem: *The primal-dual algorithm is a 2-approximation algorithm for the vertex cover problem*

Proof:

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